Spherical designs via Brouwer fixed point theorem

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Abstract

For each $N \geq c_d n^{\frac{2d(d+1)}{d+2}}$ we prove the existence of a spherical n-design on S^d consisting of N points, where c_d is a constant depending only on d.

Keywords: Spherical designs, Brouwer fixed point theorem, Marcinkiewich-Zygmund inequality, area-regular partitions.

1 Introduction

Let S^d be the unit sphere in $\mathbb{R}^{d+1}$ with normalized Lebesgue measure $d\mu_d$ $\left(\int_{S^d} d\mu_d(x) = 1\right)$. The following concept of a spherical design was introduced by Delsarte, Goethals and Seidel [5]:
A set of points $x_1, \ldots, x_N \in S^d$ is called a *spherical n-design* if

$$\int_{S^d} P(x) d\mu_d(x) = \frac{1}{N} \sum_{i=1}^{N} P(x_i)$$

for all algebraic polynomials in $d+1$ variables and of total degree at most n. For each $n \in \mathbb{N}$ denote by $N(d,n)$ the minimal number of points in a spherical n-design. The following lower bounds

$$N(d,n) \geq \binom{d+k}{d} + \binom{d+k-1}{d}, \quad n = 2k, \tag{1}$$

$$N(d,n) \geq 2\binom{d+k}{d}, \quad n = 2k+1,$$

are also proved in [5].

Spherical n-designs attaining these bounds are called tight. Exactly eight tight spherical designs are known for $d \geq 2$ and $n \geq 4$. All such configurations of points are highly symmetrical and possess other extreme properties. For example, the shortest vectors in the E_8 lattice form a tight 7-design in S^7, and a tight 11-design in S^{23} is obtained from the Leech lattice in the same way [4]. In general, lattices are a good source for spherical designs with small (d,n) [7].

On the other hand construction of spherical n-design with minimal cardinality for fixed d and $n \to \infty$ becomes a difficult analytic problem even for $d = 2$. There is a strong relation between this problem and the problem of findind N points on a sphere S^2 that minimize the energy functional

$$E(\vec{x}_1, \ldots, \vec{x}_N) = \sum_{1 \leq i < j \leq N} \frac{1}{\|\vec{x}_i - \vec{x}_j\|},$$

see Saff, Kuijlaars [12].

Let us begin by giving a short history of asymptotic upper bounds on $N(d,n)$ for fixed d and $n \to \infty$. First, Seymour and Zaslavsky [13] have proved that spherical design exists for all d, $n \in \mathbb{N}$. Then, Wagner [14] and Bajnok [2] independently proved that $N(d,n) \le c_d n^{Cd^4}$ and $N(d,n) \le c_d n^{Cd^3}$ respectively. Korevaar and Meyers have [8] improved this inequalities by showing that $N(d,n) \le c_d n^{(d^2+d)/2}$. They have also conjectured that $N(d,n) \le c_d n^d$. Note that (1) implies $N(d,n) \ge C_d n^d$. In what follows we denote by b_d, c_d, c_{1d}, etc., sufficiently large constants depending only on d. In [3] we proved the following

Theorem BV. Let a_d be the sequence defined by

$$a_1 = 1, \quad a_2 = 3, \quad a_{2d-1} = 2a_{d-1} + d, \quad a_{2d} = a_{d-1} + a_d + d + 1, \quad d \ge 2.$$

Then for all d, $n \in \mathbb{N}$,

$$N(d,n) \le c_d n^{a_d}.$$

Corollary BV. For each $d \ge 3$ and $n \in \mathbb{N}$ we have

$$N(d,n) \le c_d n^{a_d}.$$

$$a_3 \le 4, \quad a_4 \le 7, \quad a_5 \le 9, \quad a_6 \le 11, \quad a_7 \le 12, \quad a_8 \le 16, \quad a_9 \le 19, \quad a_{10} \le 22,$$

and

$$a_d < \frac{d}{2} \log_2 2d, \quad d > 10.$$

In this paper we suggest a new nonconstructive approach for obtaining new upper bounds for $N(d,n)$. We will make extensive use of the Brouwer fixed point theorem (the source of nonconstructive nature of our method), the Marcinkiewich-Zygmund inequality on the sphere [10] and the notion of area-regular partitions [9]. The main result of this paper is

Theorem 1. *For each $N \ge c_d n^{\frac{2d(d+1)}{d+2}}$ there exists a spherical n-design on S^d consisting of N points.*

This result improves our previous estimate on $N(d,n)$ for all $d > 3$, $d \neq 7$, and in particular allows us to remove the "nasty" logarithm in the power in Corollary BV, so that the function in the power has a linear behavior, which confirms the conjecture of Korevaar and Meyers. Finally, Theorem 1 guaranties the existence of spherical n-design for each N greater then our new existence bound.

2 Preliminaries

Let Δ be the Laplace operator in $\mathbb{R}^{d+1}$

$$\Delta = \sum_{j=1}^{d+1} \frac{\partial^2}{\partial x_j^2}.$$

We say that a polynomial P in $\mathbb{R}^{d+1}$ is harmonic if $\Delta P = 0$. For integer $k \geq 1$, the restriction to S^d of a homogeneous harmonic polynomial of degree k is called a spherical harmonic of degree k. The vector space of all spherical harmonics of degree k will be denoted by $\mathcal{H}_k$ (see [10] for details). The dimension of $\mathcal{H}_k$ is given by

$$\dim \mathcal{H}_k = \frac{2k+d-1}{k+d-1}\binom{d+k-1}{k}.$$

The vector spaces $\mathcal{H}_k$ are invariant under the action of the orthogonal group $O(d+1)$ on S^d and are orthogonal to each other with respect to the scalar product

$$\langle P, Q\rangle := \int_{S^d} P(x)Q(x)d\mu_d(x).$$

Another remarkable property of harmonic polynomials is that the spaces $\mathcal{H}_k$ are eigenspaces of the spherical Laplacian (Laplace-Beltrami operator [6])

$$\widetilde{\Delta} f(x) := \Delta f(\frac{x}{\|x\|}). \tag{2}$$

Thus, for a polynomial $P \in \mathcal{H}_k$ we have

$$\widetilde{\Delta} P = -k(k+d-1)P. \tag{3}$$

Here and below we use the notations $\|\cdot\|$ and $(\cdot,\cdot)$ for the Euclidean norm and usual scalar product in $\mathbb{R}^{d+1}$, respectively. For a twice differentiable function $f:\mathbb{R}^{d+1}\to\mathbb{R}$ and a point $x_0\in\mathbb{R}^{d+1}$ denote by

$$\frac{\partial f}{\partial x}(x_0) := \left(\frac{\partial f}{\partial x_1}(x_0),\ldots \frac{\partial f}{\partial x_{d+1}}(x_0)\right)$$

and

$$\frac{\partial^2 f}{\partial x^2}(x_0) := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)\right)_{i,j=1}^{d+1}$$

the gradient and the matrix of second derivatives of f (Hessian matrix) at the point x_0 respectively. Analogously to (2) we will also define for a polynomial $Q\in\mathcal{P}_n$ the spherical gradient

$$\nabla Q(x) := \frac{\partial}{\partial x}Q(\frac{x}{\|x\|})$$

and the Hessian matrix on the sphere

$$\nabla^2 Q(x) := \frac{\partial^2}{\partial x^2}Q(\frac{x}{\|x\|}). \tag{4}$$

We will also write

$$\nabla^2 Q\cdot x\cdot y := (\nabla^2 Q\cdot x, y) \qquad \text{for } x,y\in\mathbb{R}^{d+1}.$$

One consequence of Stokes's theorem is the first Green's identity [15]

$$\int_{S^d} P(x)\widetilde{\Delta}Q(x)d\mu_d(x) = -\int_{S^d}(\nabla P(x),\nabla Q(x))d\mu_d(x). \tag{5}$$

Let $\mathcal{P}_n$ be the vector space of polynomials P of degree $\leq n$ on S^d such that

$$\int_{S^d} P(x)d\mu_d(x) = 0.$$

Each polynomial in $\mathbb{R}^{d+1}$ can be written as a finite sum of terms, each of which is a product of a harmonic and a radial polynomial (i.e. a polynomial

which depends only on $\|x\|$). Therefore the vector space $\mathcal{P}_n$ decomposes into the direct sum $\mathcal{H}_k$

$$\mathcal{P}_n = \bigoplus_{k=1}^{n} \mathcal{H}_k.$$

For each vector of positive weights $w = (w_1, \ldots, w_n)$ we can define a scalar product $\langle \cdot, \cdot \rangle_w$ on $\mathcal{P}_n$ invariant with respect to the action of $O(d+1)$ on S^d by

$$\langle P, Q \rangle_w := \sum_{k=1}^{n} w_k \langle P_k, Q_k \rangle,$$

where $P_k,\ Q_k \in \mathcal{H}_k,\ P = P_1 + \ldots + P_n$ and $Q = Q_1 + \ldots + Q_n$. For each $Q \in \mathcal{P}_n$ denote by

$$\|Q\|_w = \sqrt{\langle Q, Q \rangle_w}$$

the norm corresponding to this scalar product. We will also define the operator

$$\Delta_w P := \sum_{k=1}^{n} \frac{k(k+d-1)}{w_k} P_k,\ \ P \in \mathcal{P}_n.$$

Then from (3) and (5) we get

$$\langle \Delta_w P, Q \rangle_w = \int_{S^d} \langle \nabla P(x), \nabla Q(x) \rangle d\mu_d(x). \tag{6}$$

Now, for each point $x \in S^d$ there exists a unique polynomial $G_x \in \mathcal{P}_n$ (depending on w) such that

$$\langle G_x, Q \rangle_w = Q(x) \ \text{ for all } \ Q \in \mathcal{P}_n.$$

Then, the set of points $x_1, \ldots, x_N \in S^d$ form a spherical design if and only if

$$G_{x_1} + \ldots + G_{x_N} = 0.$$

To construct the polynomials G_x explicitly we will use the Gegenbauer polynomials G_k^α [1]. For a fixed α, the G_k^α are orthogonal on $[-1, 1]$ with respect to the weight function $\omega(t) = (1-t^2)^{\alpha - \frac{1}{2}}$, that is

$$\int_{-1}^{1} G_m^\alpha(t) G_n^\alpha(t)(1-t^2)^{\alpha-\frac{1}{2}} dt = \delta_{mn} \frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(\alpha+n)\Gamma^2(\alpha)}.$$

Set $\alpha := \frac{d-1}{2}$, and let

$$G_x(y) := g_w((x, y)),$$

where

$$g_w(t) := \sum_{k=1}^{n} \frac{\dim \mathcal{H}_k}{w_k G_k^\alpha(1)} G_k^\alpha(t).$$

In order to show that $\langle P_x, Q\rangle_w = G_x(Q) = Q(x)$ for each $Q \in \mathcal{P}_n$ we will use the following identity for Gegenbauer polynomials [10]

$$G_k^\alpha((x, y)) = \frac{G_k^\alpha(1)}{\dim \mathcal{H}_k} \sum_{j=1}^{\dim \mathcal{H}_k} Y_{jk}(x) Y_{jk}(y), \tag{7}$$

where $x, y \in S^d$ and Y_{jk} are some orthonormal basis in the space $(\mathcal{H}_k, \mu_d)$. In particular, for a fixed $x \in S^d$, $G_k^\alpha((x, y)) \in \mathcal{H}_k$. Therefore, for a polynomial $Q \in \mathcal{P}_n$ we have

$$\langle G_x, Q\rangle_w = \sum_{k=1}^{n} w_i \langle G_k, Q_k\rangle = \sum_{k=1}^{n} \int_{S^d} G_k^\alpha((x, y)) Q_k(y) d\mu_d(y) =$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{\dim \mathcal{H}_k} Y_{jk}(x) \int_{S^d} Q_k(y) Y_{jk}(y) d\mu_d(y) = \sum_{k=1}^{n} Q_k(x) = Q(x).$$

Fix the weight vector $w = (w_1, \ldots, w_n)$ such that $w_k = k(k + d - 1)$. Further we will use the following additional equalities for Gegenbauer polynomials [1]:

$$G_n^\alpha(1) = \binom{2\alpha + n - 1}{n},$$

and

$$\frac{d}{dt} G_n^\alpha(t) = 2\alpha G_{n-1}^{\alpha+1}(t), \qquad \frac{d^2}{dt^2} G_n^\alpha(t) = 4\alpha(\alpha + 1) G_{n-2}^{\alpha+2}(t). \tag{8}$$

Applying Cauchy's inequality to (7) we get, for all $k \in \mathbb{N}$ and $x, y \in S^d$,

$$|G_k^\alpha((x, y))|^2 \leq G_k^\alpha((x, x)) G_k^\alpha((y, y)),$$

and hence

$$\max_{x\in[-1,1]} |g_w(x)| = g_w(1).$$

Similarly, by (8) we obtain

$$\max_{x\in[-1,1]} |g_w'(x)| = g_w'(1). \tag{9}$$

Finally, let us estimate $g_w'(1)$ and $g_w''(1)$. We have

$$g_w'(1) = \sum_{k=1}^{n} \frac{\dim \mathcal{H}_k}{w_k G_k^\alpha(1)} G_k^{\alpha\prime}(1) = \sum_{k=1}^{n} \frac{(2k+d-1)(k+d-2)!}{k!d!} \le c_{1d} n^d. \tag{10}$$

Hence, by (9) and Markov inequality we get

$$g_w''(1) < n^2 \max_{x\in[-1,1]} |g_w'(x)| = n^2 g_w'(1) \le c_{1d} n^{d+2}. \tag{11}$$

3 Proof of Theorem 1

Fix $n \in \mathbb{N}$. As mentioned in section 2, points $x_1, \ldots, x_N$ form a spherical n-design if and only if $G_{x_1} + \ldots + G_{x_N} = 0$. First we will construct a set of points such that the norm $\|G_{x_1} + \ldots + G_{x_N}\|_w$ is small, and then we will use the Brouwer fixed point theorem to show that there exists a collection of points $\{y_1, \ldots, y_N\}$ "close" to $\{x_1, \ldots, x_N\}$ with $\|G_{y_1} + \ldots + G_{y_N}\|_w = 0$.

Let $\mathcal{R} = \{R_1, \ldots, R_N\}$ be a finite collection of closed, non-overlapping (i.e., having no common interior points) regions $R_i \subset S^d$ such that $\cup_{i=1}^N R_i = S^d$. The partition $\mathcal{R}$ is called area-regular if $\mathrm{vol} R_i := \int_{R_i} d\mu_d(x) = 1/N$, for all $i = 1, \ldots, N$. The partition norm for $\mathcal{R}$ is defined by

$$\|\mathcal{R}\| := \max_{R\in\mathcal{R}} \operatorname{diam} R.$$

Now we will prove

Lemma 1. *For each $N \in \mathbb{N}$ there exists an area-regular partition $\mathcal{R} = \{R_1, \ldots, R_N\}$ of S^d and a collection of points $x_i \in R_i$, $i = 1, \ldots, N$ such that*

$$\left\| \frac{G_{x_1} + \ldots + G_{x_N}}{N} \right\|_w \le \frac{b_d n^{d/2}}{N^{1/2+1/d}}.$$

Proof. As shown in [9], for each $N \in \mathbb{N}$ there exists an area-regular partition $\mathcal{R} = \{R_1, \ldots, R_N\}$ such that $\|\mathcal{R}\| \leq c_{2d} N^{1/d}$ for some constant c_{2d}. For this partition $\mathcal{R}$ we will estimate the average value of $\left\| \frac{G_{x_1} + \ldots + G_{x_N}}{N} \right\|_w^2$, when the points x_i are uniformly distributed over R_i. We have

$$\begin{aligned}
&\frac{1}{\mathrm{vol}R_1 \cdots \mathrm{vol}R_N} \int_{R_1 \times \cdots \times R_N} \left\| \frac{G_{x_1} + \ldots + G_{x_N}}{N} \right\|_w^2 d\mu_d(x_1) \cdots d\mu_d(x_N) = \\
&\quad = \frac{1}{\mathrm{vol}R_1 \cdots \mathrm{vol}R_N} \int_{R_1 \times \cdots \times R_N} \frac{1}{N^2} \sum_{i,j=1}^{N} \langle G_{x_i}, G_{x_j} \rangle_w d\mu_d(x_1) \cdots d\mu_d(x_N) \\
&\quad = \sum_{i \neq j} \int_{R_i \times R_j} \langle G_{x_i}, G_{x_j} \rangle_w d\mu_d(x_i) d\mu_d(x_j) + \sum_{i=1}^{N} \frac{1}{N} \int_{R_i} \langle G_{x_i}, G_{x_i} \rangle_w d\mu_d(x_i) \\
&\quad = \int_{S^d \times S^d} \langle G_x, G_y \rangle_w d\mu_d(x) d\mu_d(y) + \\
&\qquad + \sum_{i=1}^{N} \left(\frac{1}{N} \int_{R_i} \langle G_x, G_x \rangle_w d\mu_d(x) - \int_{R_i \times R_i} \langle G_x, G_y \rangle_w d\mu_d(x) d\mu_d(y) \right) \\
&\quad = \int_{S^d \times S^d} g_w((x,y)) d\mu_d(x) d\mu_d(y) + \\
&\qquad + \sum_{i=1}^{N} \int_{R_i \times R_i} g_w(1) - g_w((x,y)) d\mu_d(x) d\mu_d(y).
\end{aligned}$$

The first term of the sum is equal to zero because for each fixed $x \in S^d$, the polynomial $g_w((x,y)) \in \mathcal{P}_n$. We can estimate the second term by

$$\sum_{i=1}^{N} \int_{R_i \times R_i} g_w(1) - g_w((x,y)) d\mu_d(x) d\mu_d(y) \leq \frac{1}{N} \max_{R_i \in \mathcal{R}} \max_{x,y \in R_i} |g_w(1) - g_w((x,y))|$$

$$\leq \frac{1}{N} \max_{R_i \in \mathcal{R}} \max_{x,y \in R_i} g_w'(1) \|x - y\|^2 \leq \frac{1}{N} c_{1d} n^d \|\mathcal{R}\|^2 \leq c_{1d} \frac{c_{2d}^2 n^d}{N^{1+2/d}},$$

where in the last line we use (9) and (10). This immediately implies the statement of the Lemma. □

For a polynomial $Q \in \mathcal{P}_n$ define the norm of the Hessian matrix on the sphere, as defined by (4), at the point $x_0 \in S^d$ by

$$\left\| \nabla^2 Q(x_0) \right\| = \max_{\|y\|=1} |\nabla^2 Q(x_0) \cdot y \cdot y|,$$

where the maximum is taken over vectors y orthogonal to x_0. We will prove the following estimate

Lemma 2. *For a polynomial $Q \in \mathcal{P}_n$ and point $x_0 \in S^d$*

$$\left\|\nabla^2 Q(x_0)\right\| \leq (3g_w''(1) + g_w'(1))^{1/2}\|Q\|_w.$$

Proof. Fix a unit vector y_0 orthogonal to x_0 and define a curve $x(t)$ on the sphere S^d by

$$x(t) = x_0 \cos(t) + y_0 \sin(t).$$

For each $t \in \mathbb{R}$ we consider the polynomial $G_{x(t)}(y) = g_w((x(t), y)) \in \mathcal{P}_n$, which has the property $\langle Q, G_{x(t)}\rangle_w = Q(x(t))$ for all $Q \in \mathcal{P}_n$. Setting $G'' = \frac{d^2}{dt^2}Gx(t)|_{t=0}$, we have that

$$\nabla^2 Q(x_0) \cdot y_0 \cdot y_0 = \frac{d^2}{dt^2}Q(x(t))|_{t=0} = \langle Q, G''\rangle_w. \tag{12}$$

Hence

$$\left\|\nabla^2 Q(x_0)\right\| \leq \|G''\|_w \|Q\|_w.$$

It remains to show that $\|G''\|_w = (3g_w''(1) + g_w'(1))^{1/2}$. Since

$$\frac{d^2}{dt^2}G_{x(t)}(y) = \frac{d^2}{dt^2}g_w((x(t), y)),$$

we obtain

$$G''(y) = (y_0, y)^2 g_w''((x_0, y)) - (x_0, y)g_w'((x_0, y)). \tag{13}$$

From (12) and (13) we get by direct calculation

$$\langle G'', G''\rangle_w = \frac{d^2}{dt^2}G''(x(t))|_{t=0} = 3g_w''(1) + g_w'(1).$$

Lemma 2 is proved. $\square$

Denote by B^q the closed ball of radius 1 with center at 0 in $\mathbb{R}^q$. To prove the following Lemma 3 we use the Brouwer fixed point theorem [11]

Theorem B. Let A be a closed bounded convex subset of $\mathbb{R}^q$ and $H : A \to A$ be a continuous mapping on A. Then there exists some $z \in A$ such that $H(z) = z$.

Lemma 3. *Let $F : B^q \to \mathbb{R}^q$ be a continuous map such that*

$$F(x) = A(x) + G(x),$$

where $A(x)$ is a linear map and for each $x \in B^q$

$$\|A(x)\| \geq \alpha \|x\| \tag{14}$$

and

$$\|G(x)\| \leq \alpha \|x\|/2, \tag{15}$$

for some $\alpha > 0$. Then, the image of F contains the closed ball of radius $\alpha/2$ with center at 0.

Proof. Take an arbitrary y, with $\|y\| \leq \alpha/2$. It is sufficient to show that there exists $x \in B^q$ such that $F(x) = y$. The inequality (14) implies that $\|A^{-1}(y)\| \leq 1/2$. Denote by K the ball of radius $1/2$ with center 0. Consider a map

$$H_y(z) = -A^{-1}(G(A^{-1}(y) + z)).$$

By (14) and (15) we obtain that $H_y(K) \subset K$. Hence, by the Brouwer fixed point theorem, there exists $z \in K$ such that $H_y(z) = z$. This then implies that

$$F(A^{-1}(y) + z) = y.$$

□

To prove the principal Lemma 4 we also need a result which is an easy corollary of Theorem 3.1 in [10]

Theorem MNW. There exist constants r_d and N_d such that for each area-regular partition $\mathcal{R} = \{R_1, \ldots, R_N\}$ with $\|\mathcal{R}\| < \frac{r_d}{m}$, each collection of points $x_i \in R_i$, $i = 1, \ldots, N$ and each algebraic polynomial P of total degree $m > N_d$ the following inequality

$$\frac{1}{2} \int_{S^d} |P(x)| d\mu_d(x) < \frac{1}{N} \sum_{i=1}^{N} |P(x_i)| < \frac{3}{2} \int_{S^d} |P(x)| d\mu_d(x) \tag{16}$$

holds.

Consider the map $\Phi : (S^d)^N \to \mathcal{P}_n$ defined by

$$(x_1, \dots, x_N) \xrightarrow{\Phi} \frac{G_{x_1}+\ldots+G_{x_N}}{N} .$$

Lemma 4. *Let $x_1, \dots, x_N \in S^d$ be the collection of points and $\mathcal{R} = \{R_1, \dots, R_N\}$ an area-regular partition such that $x_i \in R_i$ and $\|\mathcal{R}\| \le \frac{r_d}{2n}$. Then the image of the map Φ contains a ball of radius $\rho \ge A_d n^{(-d-2)/2}$ with center at the point $G = \frac{G_{x_1}+\ldots+G_{x_N}}{N}$, where A_d is a sufficiently small constant, depending only on d.*

Proof. For each polynomial $P \in \mathcal{P}_n$ consider the circles on S^d given by

$$\tilde{x}_i(t) = x_i \cos(\|\nabla P(x_i)\| t) + y_i \sin(\|\nabla P(x_i)\| t),$$

where $y_i = \frac{\nabla P(x_i)}{\|\nabla P(x_i)\|}$, $i = \overline{1, \dots, N}$. Define the map $X : \mathcal{P}_n \to (S^d)^N$ by

$$X(P) = (x_1(P), \dots, x_N(P)) := (\tilde{x}_1(1), \dots, \tilde{x}_N(1)).$$

Now we will consider the composition $L = \Phi \circ X : \mathcal{P}_n \to \mathcal{P}_n$ which takes the form

$$L(P) = \frac{G_{x_1(P)} + \ldots + G_{x_N(P)}}{N}.$$

For each $Q \in \mathcal{P}_n$ one can take the Taylor expansion

(17)

$$\langle G_{\tilde{x}_i(t)}, Q\rangle_w = Q(\tilde{x}_i(t)) = Q(x_i) + \frac{d}{dt}Q(\tilde{x}_i(0))t + \frac{1}{2}\cdot\frac{d^2}{dt^2}Q(\tilde{x}_i(t_i))t^2, \quad t_i \in [0, t].$$

Hence, we can represent the function $L(P)$ in the form

$$L(P) = L(0) + L'(P) + L''(P).$$

Here $L'(P)$ is the unique polynomial in $\mathcal{P}_n$ satisfying

$$\langle L'(P), Q\rangle_w = \frac{1}{N}\sum_{i=1}^{N}(\nabla Q(x_i), \nabla P(x_i)) \ \text{ for all } \ Q \in \mathcal{P}_n,$$

and

$$L''(P) = L(P) - L(0) - L'(P).$$

First, for each $P \in \mathcal{P}_n$ we will estimate the norm of $L'(P)$ from below. We have

$$\|L'(P)\|_w \geq \frac{1}{\|P\|_w} \cdot \langle L'(P), P\rangle_w = \frac{1}{\|P\|_w} \cdot \frac{1}{N}\sum_{i=1}^{N}(\nabla P(x_i), \nabla P(x_i)).$$

Applying (16) to the polynomial $(\nabla P, \nabla P)$ of degree $\leq 2n$, we get

$$\frac{1}{N}\sum_{i=1}^{N}(\nabla P(x_i), \nabla P(x_i)) \geq \frac{1}{2}\int_{S^d}(\nabla P(x), \nabla P(x))d\mu_d(x).$$

On the other hand, by (6) we have

$$\int_{S^d}(\nabla P(x), \nabla P(x))d\mu_d(x) = \langle P, \Delta_w P\rangle_w = \|P\|_w^2.$$

This gives us the estimate

$$\|L'(P)\|_w \geq \frac{1}{2}\|P\|_w. \tag{18}$$

Now we will estimate the norm of $L''(P)$ from above. By (17) we have

$$\langle L''(P), Q\rangle_w = \frac{1}{2N}\sum_{i=1}^{N}\frac{d^2}{dt^2}Q(\tilde{x}_i(t_i)),$$

for some $t_i \in [0, 1]$. Since the following equality holds

$$\frac{d^2}{dt^2}Q(\tilde{x}_i(t)) = \nabla^2 Q \cdot \frac{d\tilde{x}_i(t)}{dt} \cdot \frac{d\tilde{x}_i(t)}{dt},$$

Lemma 2 implies that

$$|\frac{d^2}{dt^2}Q(\tilde{x}_i(t))| \leq (3g_w''(1) + g_w'(1))^{1/2}\|\frac{d\tilde{x}_i}{dt}\|^2 \cdot \|Q\|_w.$$

It follows from the identity

$$\|\frac{d\tilde{x}_i}{dt}(t)\| = \|\nabla P(x_i)\|$$

and estimates (10), (11) that

$$|\frac{d^2}{dt^2}Q(\tilde{x}_i(t))| \le c_{3d}n^{(d+2)/2}\|\nabla P(x_i)\|^2 \cdot \|Q\|_w.$$

This inequality yields immediately

$$|\langle L''(P), Q\rangle_w| = |\frac{1}{2N}\sum_{i=1}^{N}\frac{d^2}{dt^2}Q(\tilde{x}_i(t_i))| \le \frac{c_{3d}n^{(d+2)/2}\|Q\|_w}{N}\sum_{i=1}^{N}\|\nabla P(x_i)\|^2.$$

Applying again (16), we obtain

$$\frac{1}{N}\sum_{i=1}^{N}\|\nabla P(x_i)\|^2 \le \frac{3}{2}\|P\|_w^2.$$

So, for each $Q \in \mathcal{P}_n$ we have that

$$|\langle L''(P), Q\rangle_w| \le \frac{3}{2}c_{3d}n^{(d+2)/2}\|P\|_w^2 \cdot \|Q\|_w.$$

Thus, we get

$$\|L''(P)\|_w \le \frac{3}{2}c_{3d}n^{(d+2)/2}\|P\|_w^2. \tag{19}$$

Lemma 3 combined with inequalities (18) and (19) implies that the image of L, and hence the image of Φ, contains a ball of radius $\rho \ge A_d n^{(-d-2)/2}$ around $L(0) = G$, where $A_d = 1/6c_{3d}$, proving the lemma. □

Proof of Theorem 1. By Lemma 1, there exists an area-regular partition $\mathcal{R} = \{R_1, \dots, R_N\}$ such that $\|\mathcal{R}\| \le c_{2d}N^{1/d}$, and a collection of points $x_i \in R_i$, $i = 1, \dots, N$ such that

$$\left\|\frac{G_{x_1} + \ldots + G_{x_N}}{N}\right\|_w \le \frac{b_d n^{d/2}}{N^{1/2+1/d}}.$$

Take N large enough such that $N > N_d$ and $\frac{c_{2d}}{N^{1/d}} < \frac{r_d}{2n}$, where N_d and r_d are defined by Theorem MNW. Applying Lemma 4 to the partition $\mathcal{R}$ and the collection of points $x_1, \dots, x_N$, we obtain immediately that $G_{y_1} + \ldots + G_{y_N} = 0$ for some $y_1, \dots, y_N \in S^d$ if

$$\frac{b_d n^{d/2}}{N^{1/2+1/d}} < A_d n^{(-d-2)/2}.$$

So, we can choose a constant c_d such that the last inequality holds for all $N > c_d n^{\frac{2d(d+1)}{d+2}}$. Theorem 1 is proved. □

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